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GENERALIZED GEOMETRIC LINNIK DISTRIBUTION

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ABSTRACT
In this paper, we introduce and study type II generalized geometric Linnik ($GeGL_2$) distribution. A representation of $GeGL_2$ distribution is obatined. It is shown that $GeGL_2$ distribution arises as the limit distribution of negative binomial sum of iid generalized Linnik random variables. $GeGL_2$ stochastic process is introduced and studied.

KEYWORDS
Geometric gamma process, Geometric Linnik Distribution, Generalized Geometric Linnik Distribution, Linnik distribution, Stable Laws

1. Introduction

As a generalization of the Linnik distribution [10] introduced semi α-Laplace distribution. A random variable X on R has semi α-Laplace distribution if its characteristic function $\phi(t)$ is of the form

$$\phi(t) = \frac{1}{1 + |t|^{\alpha}\delta(t)} \tag{1}$$

where $\delta(t)$ satisfies the functional equation

$$\delta(t) = \delta(p^{1/\alpha}t), 0 < p < 1, 0 < \alpha \leq 2. \tag{2}$$

[3] derived expression for the density function of α-Laplace random variables in terms of Meijer's G-function and obtained a multivariate generalization of α-Laplace distribution. [9] introduced generalized Linnik law with characteristic function

$$\phi(t) = \frac{1}{(1 + |t|^{\alpha})^{v}}, v > 0, 0 < \alpha \leq 2. \tag{3}$$

This distribution is known as Pakes generalized Linnik distribution. When $v = 1$, it reduces to α-Laplace distribution where as when $\alpha = 2$, it reduces to the general-

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ized Laplacian distribution of [8]. [6] developed Pakes generalized Linnik first order autoregressive process and studied its propoerties.

Definition 1.1. A random variable X on R is said to have geometric Linnik distribution and write $X \overset{d}{=} GL(\alpha, \lambda)$ if its characteristic function $\phi(t)$ is

$$\phi(t) = \frac{1}{1 + ln(1 + \lambda|t|^{\alpha})}, t \in R, 0 < \alpha \leq 2, \lambda > 0 \tag{4}$$

Definition 1.2. A random variable X on R is said to have generalized Linnik distribution and write $X \overset{d}{=} GeL(\alpha, \lambda, p)$ if has the characteristic function

$$\phi(t) = \frac{1}{(1 + \lambda|t|^{\alpha})^p}, p > 0, \lambda > 0, 0 < \alpha \leq 2 \tag{5}$$

[1]& [2] considered a generalization of $GeL(\alpha, \lambda, p)$ distribution and studied its properties. They discussed the analytic and asymptotic properties of this distribution and obtained some integral and series representation of its probability density.
[7] studied some properties of Geometric Linnik distribution and estimated the parameters of Geometric Linnik distribution using empirical characteristic function. Type I generalized geometric Linnik distribution is studied in this paper. In Section 2, another generalization of geometric Linnik distribution is introduced and the properties of this type II Generalized Geometric Linnik ($GeGL_2$) distribution are studied. A representation of $GeGL_2$ distribution is obatined. It is shown that $GeGL_2$ distribution arises as the limit distribution of negative binomial sum of iid generalized Linnik random variables. $GeGL_2$ stochastic process is introduced and studied. Possible applications are also discussed.

2. GENERALIZED GEOMETRIC LINNIK DISTRIBUTION

Definition 2.1. A random variable X on R is said to have type I generalized geometric Linnik distribution and write $X \overset{d}{=} GeGL_1(\alpha, \lambda, p)$ if has the characteristic function

$$\phi(t) = \frac{1}{1 + p \ ln(1 + \lambda|t|^{\alpha})}, p > 0, \lambda > 0, 0 < \alpha \leq 2 \tag{6}$$

[6] called the distribution having characteristic function (6) as geometric Pakes generalized Linnik distribution and showed that it is the limit distribution of geometric sums of Pakes generalized Linnik random variables. Here we introduce type II Generalized Geometric Linnik distribution and study its properties.

Definition 2.2. A random variable X on R is said to have type II Generalized Geometric Linnik distribution and write $X \overset{d}{=} GeGL_2(\alpha, \lambda, \tau)$ if has the characteristic function

$$\phi(t) = \left[\frac{1}{1 + ln(1 + \lambda|t|^{\alpha})}\right]^{\tau}, t \in R, \lambda, \tau > 0, 0 < \alpha \leq 2. \tag{7}$$

Note that when $\tau = 1$ type II Generalized Geometric Linnik distribution reduces to geometric Linnik distribution.

Definition 2.3. A random variable X on (0,∞) has geometric gamma distribution if it has Laplace transform

$$\phi_1(\delta) = \left[\frac{1}{1+ln(1+\delta)}\right]^{\tau}, \delta > 0, \tau > 0 \tag{8}$$

For properties of geometric gamma distribution, see [5].

A representation of type II Generalized Geometric Linnik random variable in terms of geometric gamma and stable random variable is presented below.

Theorem 2.4. *Let X and Y be independent random variables such that X has geometric gamma distribution with Laplace transform $\left[\frac{1}{1+ln(1+\delta)}\right]^{\tau}$ and Y be stable with characteristic function $e^{-\lambda|t|^{\alpha}}, 0 < \alpha \leq 2, \lambda > 0$. Then $X^{1/\alpha}Y \stackrel{d}{=} GeGL_2(\alpha, \lambda, \tau)$*

Proof.

$$\begin{aligned}
\phi_{X^{1/\alpha}Y}(t) &= E(e^{itX^{1/\alpha}Y}) \\
&= \int_0^{\infty} E(e^{itX^{1/\alpha}Y}|X = x)dF(x) \\
&= \int_0^{\infty} \phi_Y(tx^{1/\alpha})dF(x) \\
&= \int_0^{\infty} e^{-\lambda|t|^{\alpha}x}dF(x) \\
&= \left[\frac{1}{1+ln(1+\lambda|t|^{\alpha})}\right]^{\tau}.
\end{aligned} \tag{9}$$

□

Now we shall consider a limiting property of the type II generalized Linnik distribution.

Theorem 2.5. *Let $X_1, X_2, ...$ be independent and identically distributed random variables with characteristic function $\left(\frac{1}{1+\lambda|t|^{\alpha}}\right)^{1/n}$ and N be a negative binomial random variable with probability generating function $\left(\frac{pz}{1-qz}\right)^{v}, v > 0, p = \frac{1}{n}, q = 1 - p$. Then $X_1 + X_2 + ... + X_N$ converges in distribution to Z where $Z \stackrel{d}{=} GeGL_2$.*

Proof. Let $S_N = X_1 + X_2 + ... + X_N$.

$$\begin{aligned}
\phi_{S_N}(t) =& E(e^{it(X_1+X_2+...+X_N)}) \\
=& \sum_{n=1}^{\infty}[\phi(t)]^n P(N = n)
\end{aligned} \tag{10}$$

Therefore,

$$\begin{aligned}
\phi_{S_N}(t) &= \left[\frac{p\phi(t)}{1-q\phi(t)}\right]^v, where\ \phi(t) = \left(\frac{1}{1+\lambda|t|^\alpha}\right)^{1/n} \\
&= \left[\frac{p}{(1+\lambda|t|^\alpha)^{1/n}-q}\right]^v \\
&= \left[\frac{1/n}{(1+\lambda|t|^\alpha)^{1/n}-(1-\frac{1}{n})}\right]^v \\
&= \left[\frac{1}{n(1+\lambda|t|^\alpha)^{1/n}-(n-1)}\right]^v \\
&= \left[\frac{1}{1+n\left[(1+\lambda|t|^\alpha)^{1/n}-1\right]}\right]^v
\end{aligned} \tag{11}$$

Let $\phi_n(t) = \phi_{S_N}(t)$.

$$\begin{aligned}
\lim_{n\to\infty}\phi_n(t) &= \left[\frac{1}{1+\lim_{n\to\infty}\left\{n\left[(1+\lambda|t|^\alpha)^{1/n}-1\right]\right\}}\right]^v \\
&= \left[\frac{1}{1+ln(1+\lambda|t|^\alpha)}\right]^v
\end{aligned} \tag{12}$$

□

Theorem 2.6. *The function* $\phi(t) = \left[\frac{1}{1+ln(1+\lambda|t|^\alpha)}\right]^v$ *on R is a characteristic function if and only if* $0 < \alpha \le 2$ *and* $v > 0$.

Proof. Suppose for some $\alpha > 0$ the function $\phi(t)$ is a characteristic function. Then we have to prove that $\alpha \in (0, 2]$. The case $\alpha < 0$ is impossible due to the requirement that $\lim_{t\to 0}\phi(t) = 1$ for the characteristic function ϕ. Note that for each positive integer n, the function $\phi_n(t) = \left[\frac{1}{1+\frac{1}{n}ln(1+\lambda|t|^\alpha)}\right]^n$ is also a characteristic function. Let F_n denote the distribution function with characteristic function ϕ_n. Then F_n converges weakly to a Linnik characteristic function $\phi(t) = \frac{1}{1+\lambda|t|^\alpha}$. This implies that $\alpha \in (0, 2]$.

For fixed $\alpha \in (0, 2]$, the function $\frac{1}{1+\lambda|t|^\alpha}$ is the characteristic function of Linnik distribution. Hence the proof. □

Theorem 2.7. $GeGL_2(\alpha, \lambda, v)$ *is normally attracted to stable law.*

Proof. The characteristic function of $n^{-1/\alpha}(X_1 + X_2 + ... + X_n)$ is

$$\begin{aligned}
\phi_{n^{-1/\alpha}S_n}(t) &= \phi_{\frac{X_1+X_2+...+X_n}{n^{1/\alpha}}}(t) \\
&= \phi_{X_1+X_2+...+X_n}(t/n^{1/\alpha}) \\
&= [\phi_{X_1}(t/n^{1/\alpha})]^n \\
&= \left[\frac{1}{1 + ln(1 + \frac{\lambda|t|^\alpha}{n})}\right]^{vn} \\
&= \left[\frac{1}{1 + \frac{\lambda|t|^\alpha}{n} + o(\frac{1}{n^2})}\right]^{vn} \to e^{-\lambda v|t|^\alpha} \ as\ n \to \infty
\end{aligned} \tag{13}$$

This completes the proof. □

Theorem 2.8. *The generalized geometric Linnik stochastic process admits the representation* $X_{\alpha,\lambda,v}(s) \stackrel{d}{=} Y_{\alpha,\lambda} Z_{s,v}^{1/\alpha}$ *where* $Y_{\alpha,\lambda}$ *is symmetric stable with characteristic function* $e^{-\lambda|t|^\alpha}$ *and* $Z_{s,v}$ *is geometric gamma process with Laplace transform* $\left[\frac{1}{1+ln(1+\delta)}\right]^{vs}$.

Proof.

$$\begin{aligned}
\phi_{Y_{\alpha,\lambda}Z_{s,v}^{1/\alpha}}(t) &= E(e^{itYZ^{1/\alpha}}) \\
&= \int_0^\infty E(e^{itYZ^{1/\alpha}}/Z = z)dF(z) \\
&= \int_0^\infty \phi_Y(tz^{1/\alpha})dF(z) \\
&= \int_0^\infty e^{-\lambda z|t|^\alpha} dF(z) \\
&= \left[\frac{1}{1 + ln(1 + \lambda|t|^\alpha)}\right]^{vs}
\end{aligned} \tag{14}$$

Therefore $X_{\alpha,\lambda,v}(s) \stackrel{d}{=} Y_{\alpha,\lambda} Z_{s,v}^{1/\alpha}$ □

[6] discussed the applications of Pakes generalized Linnik distribution and geometric Pakes generalized Linnik distribution. Applications of random summation in Marketing, Insurance Mathematics and Risk Theory, Reliability Theory, etc. are presented in [4]. The $GeGL_2$ model developed in this paper can be used for modelling stock price returns, speech waves etc., as an alternative to generalized Linnik laws and Pakes generalized Linnik laws.

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References

[1] Erdogan, M.B. and Ostrovskii, I.V. (1997) Non-symmetric Linnik distributions. C.R.Acad.Sci. Paris t. 325, Serie 1, 511-516.

[2] Erdogan, M.B. and Ostrovskii, I.V. (1998) Analytic and asymptotic properties of generalized Linnik probability densities. J. Math. Anal. Appl. 217, 555-579.

[3] George, S. and Pillai, R.N. (1987) Multivariate α-Laplace distributions. J. Na. Acad. of Math. 5, 13-18.

[4] Gnedenko, B.V. and Korolev, Yu.V. (1996) Random summation: Limit Theorems and Applications. CRC press, Boca Raton

[5] Jose, K.K. and Lekshmi, V.S. (1999) On geometric exponential distribution and its applications. J. Ind. Statist. Assoc. 37, 51-58.

[6] Lekshmi, V.S. and Jose, K.K. (2006) Autoregressive process with Pakes and geometric Pakes generalized Linnik marginals. Statist. Prob. Letters 76, 318-326

[7] Mariamma, A.(2013) Geometric Linnik Distribution. JP. J. Fund. Appl. Statist. 3, 1-9.

[8] Mathai, A.M. (1993) On non-central generalized Laplacianness of quadratic forms in normal variables. J. Multi. Anal. 45, 239-246.

[9] Pakes, A.G. (1998) Mixture representations for symmetric generalized Linnik laws. Statist. Prob. Letters 37, 213-221.

[10] Pillai, R.N. (1985). Semi α-Laplace distributions. Commun. Statist.- Theor. Meth. 14, 991-1000